Exercise 18

In Exercises 17–24, find the unknown if the solution of each equation is given:

$$\text{If } u(x) = e^{2x} \text{ is a solution of } u(x) = e^{2x} - \alpha(e^2+1)x + \int_0^1 xtu(t)\,dt, \text{ find } \alpha$$

Solution

Substitute the solution into both sides of the equation.

$$e^{2x} = e^{2x} - \alpha(e^2+1)x + \int_0^1 xte^{2t}\,dt$$

Subtract e^{2x} from both sides.

$$0 = -\alpha(e^2+1)x + \int_0^1 xte^{2t}\,dt$$

Bring the term with α to the left side.

$$\alpha(e^2+1)x = \int_0^1 xte^{2t}\,dt$$

x doesn't depend on t, so it can be brought in front of the integral. Also, introduce a new variable s in the exponent.

$$\begin{aligned}
&= x\int_0^1 te^{st}\,dt\bigg|_{s=2} \\
&= x\int_0^1 \frac{\partial}{\partial s}e^{st}\,dt\bigg|_{s=2} \\
&= x\frac{d}{ds}\int_0^1 e^{st}\,dt\bigg|_{s=2} \\
&= x\frac{d}{ds}\left(\frac{1}{s}e^{st}\bigg|_0^1\right)\bigg|_{s=2} \\
&= x\frac{d}{ds}\left[\frac{1}{s}(e^s-1)\right]\bigg|_{s=2} \\
&= x\left[-\frac{1}{s^2}(e^s-1)+\frac{1}{s}e^s\right]\bigg|_{s=2} \\
\alpha(e^2+1)x &= x\left[-\frac{1}{4}(e^2-1)+\frac{1}{2}e^2\right]
\end{aligned}$$

Divide both sides by x and expand the right side.

$$\begin{aligned}
\alpha(e^2+1) &= -\frac{1}{4}e^2 + \frac{1}{4} + \frac{1}{2}e^2 \\
&= \frac{1}{4}(e^2+1)
\end{aligned}$$

Therefore,

$$\alpha = \frac{1}{4}.$$